On solvability of the Cauchy problem for a second order parabolic equation degenerating into Schrodinger type

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Abstract

The Cauchy problem is investigeted for the parabolic type in the some finite part $[t_0, t_1] \subset [0, \infty)$ of the semi axis $t \in [0, \infty)$ and degenarated to Schrodinger type in the remain part of the same semi axes the second order parabolic equation.

The existence of the solution is proved under some conditions on the data and the explicit integral representation is constructed

In the semi plane $\Pi = \{(t,x)/t > 0; -\infty < x < \infty\}$ we consider the following Cauchy problem

$$p(t)\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t), \qquad (t,x) \in \Pi, \tag{1}$$

$$\lim_{t \to +0} u(t,x) = \varphi(x), \qquad -\infty < x < \infty \tag{2}$$

where $p(t), f(t,x)$ and $\varphi(x)$ are the known functions, $u = u(t,x)$ -are the desired complex valued functions.

Relatively to the coeffisients and right hand sides of problem (1), (2) it is assumed the fulfilment of the following conditions:

1^0. $p(t) \in C\,[0, \infty)\,,$

2^0. $p(t) \neq 0,$ at $t \in [0, \infty)\,,$

3^0. $Re\, p(t) \geq 0,$ at $t \in [0, \infty)$ and $Re\ p(0) > 0,$

4^0. There exists $p_0 = const > 0$ such that $\int\limits_0^t Jmp^{-1}(\tau)\ d\tau \leq p_0,$

5^0. The function $\varphi(x)$ is continuous and bounded at $x \in (-\infty, \infty)$

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6^0. *f(t,x)* is continions and bounded in the layer $\Pi'(t_0,T) = \{t_0 \le t \le T;\ -\infty < x < \infty\}$

7^0. *f(t,x)* satisfies in $\Pi'(t_0,T)$ the Holder condition wish respect to x i.e. there exist the constants B and $0 < \alpha \le 1$ such that $|f(t,x) - f(t,y)| \le B|x-y|^\alpha$ for any $(t,x),(t,y) \in \Pi'(t_0,T)$.

The formal solution of problems (1), (2) is constructed the help of method of integral Fourier transform and is represented in the form of

$$u(t,x) = \int\limits_{-\infty}^{\infty} Q(t,y-x)\varphi(y)\,dy + \int\limits_0^t \int\limits_{-\infty}^{\infty} Q_0(t-\tau,y-x)f(\tau,y)\,d\tau\,dy \qquad (3)$$

where

$$Q(t,y-x) = \frac{e^{-\frac{(y-x)^2}{4\omega(t)}}}{2\sqrt{\pi\omega(t)}}, \qquad Q(t-\tau,y-x) = \frac{e^{-\frac{(y-x)^2}{4\omega_0(t,\tau)}}}{2\sqrt{\pi\omega_0(t,\tau)}}$$

$$\omega(t) = \int\limits_0^t p^{-1}(\eta)\,d\eta, \quad \omega_0(t,\tau) = \int\limits_\tau^t p^{-1}(\eta)\,d\eta.$$

Note that equation (1) in some part of the considered interval $t \in [0,\infty)$, belongs to the parabolic type, in the other parts of interval is degenerated in Schrodinger type [1].

Let $[t_0,t_1] \subset [0,\infty)$ be a segment, where the condition

$$Re\ p(\tau) > 0, \qquad \tau \in [t_0,t_1] \qquad (4)$$

is satisfied.

At fulfilment of conditions $1^0 - 4^0$, some estimetes for the elements of integral (3) which provide uniform convergence of this integral, are obtained.

The sollowing one is proved.

Lemma 1. Let conditions $1^0 - 4^0$ be fulfilled and inepualihes (4) fold. Then the following estimete is valid

$$Re\left(\int\limits_\tau^t p^{-1}(\eta)\,d\eta\right) \le (t-\tau)\,|H(t,\tau)|\cos\arg H(t,\tau) \le (t-\tau)\,|H(t,\tau)|\sin\delta, \qquad (5)$$

here

$$|\arg H(t,\tau)| \le \frac{\pi}{2} - \delta, \;\; 0 < \delta < \frac{\pi}{2}, \;\; t \ge \tau, \;\; H(t,\tau) = \frac{1}{t-\tau}\int\limits_{\tau}^{t} P^{-1}(\eta)\, d\eta$$

Lemma 2. Let conditions $1^0 - 4^0$ be fulfilled for some $\tau_0 \in [0,\infty)$, $Re\, p(\tau_0) > 0$ then the estimate

$$Re\left(\int\limits_{\tau_0}^{t} p^{-1}(\eta)\, d\eta\right) \le (t-\tau_0)\, |H(t,\tau_0)| \sin\delta \tag{6}$$

where

$$0 < \delta < \frac{\pi}{2}, \;\; |H(t,\tau_0)| > 0$$

Lemma 3. Let conditions $1^0 - 4^0$ be fulfilled . Then the estimate

$$Re\left(\int\limits_{0}^{t} p^{-1}(\eta)\, d\eta\right) \le t\, |H_1(t)| \sin\delta \tag{7}$$

where

$$0 < \delta < \frac{\pi}{2}, \quad H_1(t) = \frac{1}{t}\int\limits_{0}^{t} p^{-1}(\eta)\, d\eta$$

is valid.

Theorem. Let conditions $5^0, 6^0, 7^0$ and the conditions of lemmas 1, 2, 3 be fulfilled. Then problem (1), (2) has a classical solution belonging to the space $C^{1,2}(t > 0,\; x \in (-\infty,\infty)) \frown C(t \ge 0, \;\; x \in (-\infty,\infty))$ and this solution is represented by formula (3).

References

[1] Gelfand I.M., Shilov G.F. Some problems of theory of differential equations (generalized functions, Issue 3), Moscow., Phizmatgiz, 1958, (Russia).